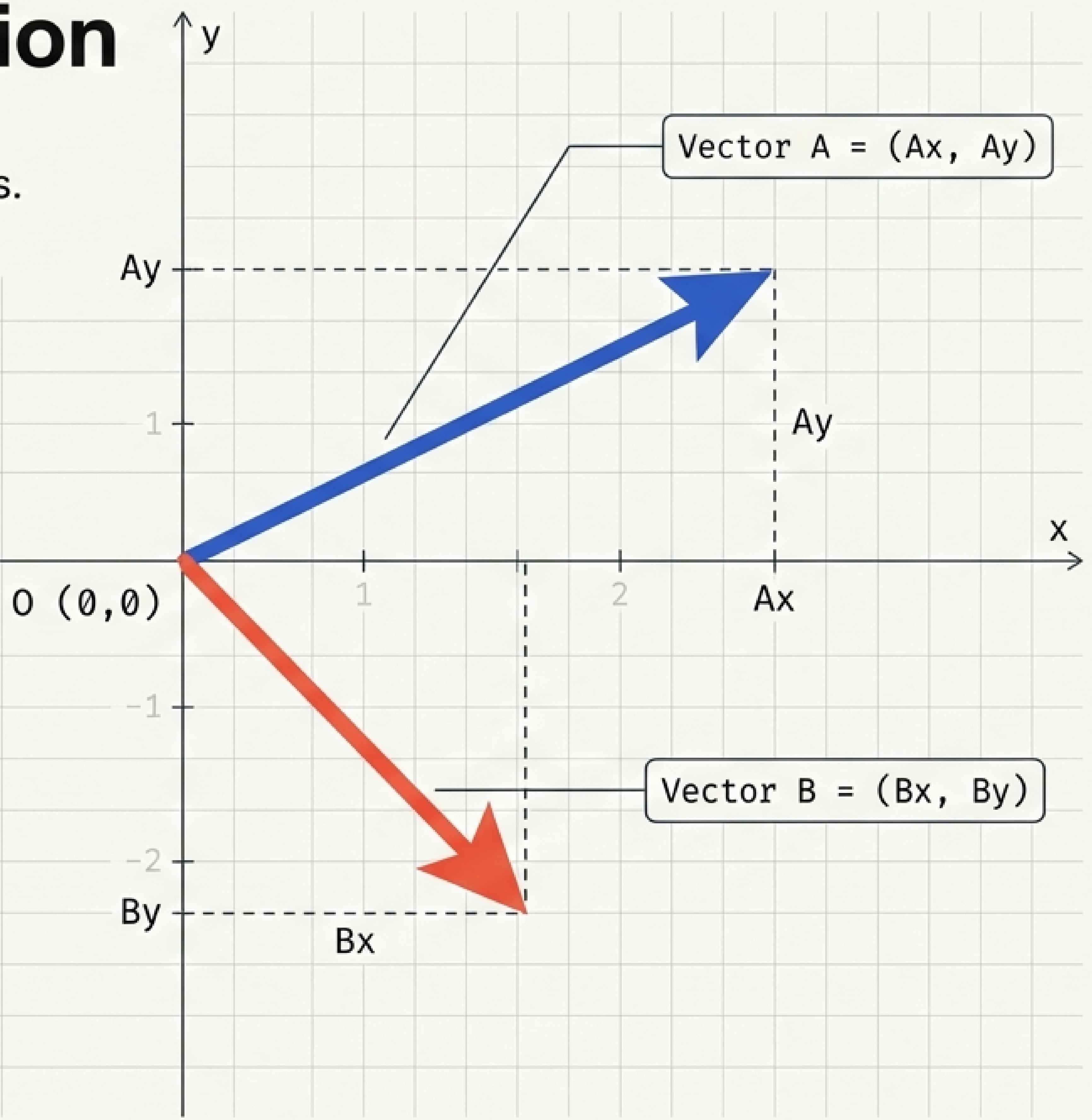
The Geometry of Motion

A masterclass in 2D kinematics: mapping magnitude, direction, and position using vectors.
Inter

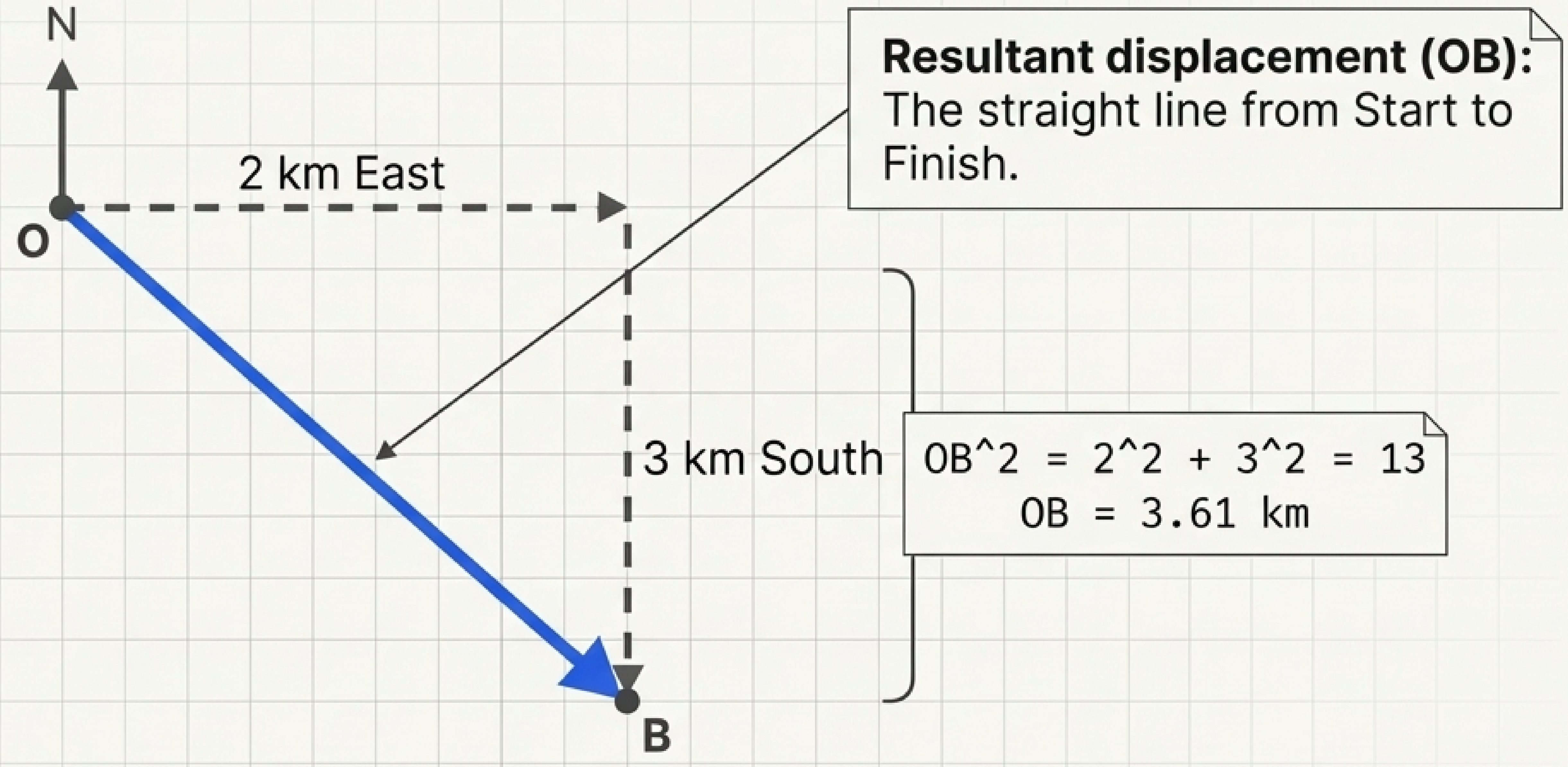


The Mechanics Matrix: Magnitude meets Direction

A scalar answers "How much?"
A vector answers "How much, and which way?"

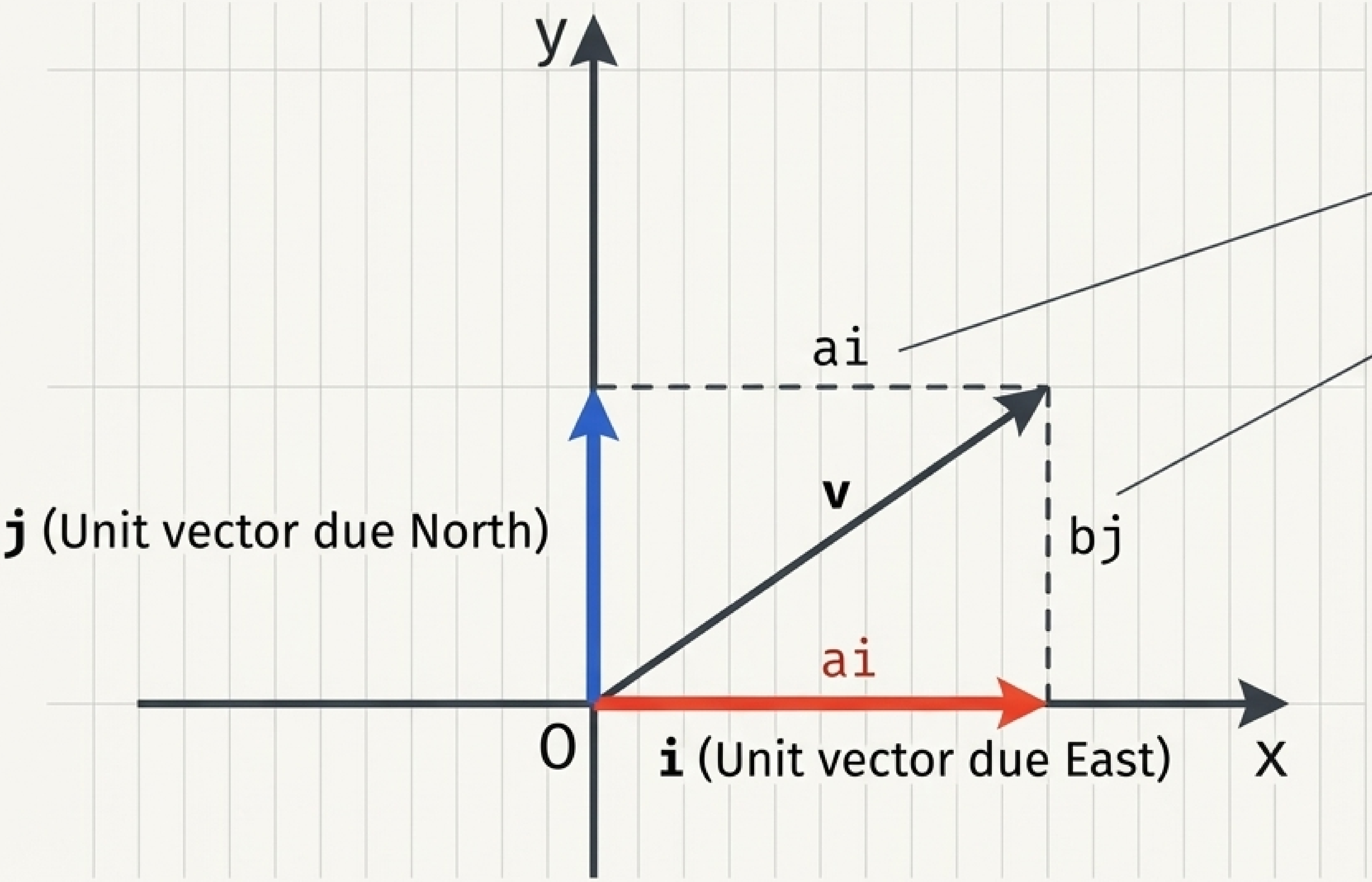
SCALARS **Magnitude Only** (Always positive)	VECTORS **Magnitude + Direction** (Can be positive, negative, or act at angles)
• Distance (m)	→ Displacement (m)
• Speed (m s-1)	↓ Velocity (m s-1)
• Time (s)	→ Acceleration (m s-2)
• Mass (kg)	→ Force / Weight (N)

Combining movements requires the Triangle Law



The distance walked is 5 km. The displacement is 3.61 km. They are not the same.

The Cartesian Language defines space in two dimensions



$\mathbf{v} = a\mathbf{i} + b\mathbf{j}$

Any vector can also be written as a column vector:

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

Vector math follows simple rules of grouping

Given: $\mathbf{p} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{q} = 5\mathbf{i} + \mathbf{j}$

1. Addition

$$\mathbf{p} + \mathbf{q} = (2\mathbf{i} + 5\mathbf{i}) + (3\mathbf{j} + 1\mathbf{j})$$

$$= 7\mathbf{i} + 4\mathbf{j}$$

Add $\mathbf{i}$ terms together, add $\mathbf{j}$ terms together.

2. Subtraction

$\mathbf{a} = 5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j}$

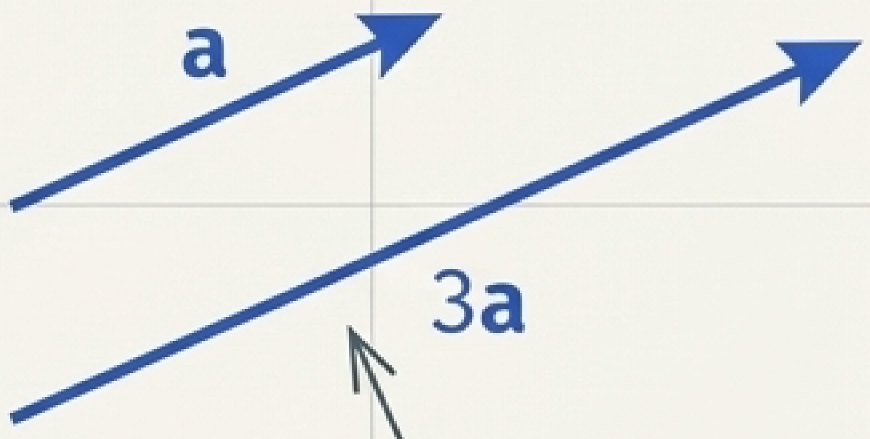
$$2\mathbf{a} = 10\mathbf{i} + 4\mathbf{j}$$

$$2\mathbf{a} - \mathbf{b} = (10\mathbf{i} - 3\mathbf{i}) + (4\mathbf{j} - (-4\mathbf{j}))$$

$$= 7\mathbf{i} + 8\mathbf{j}$$

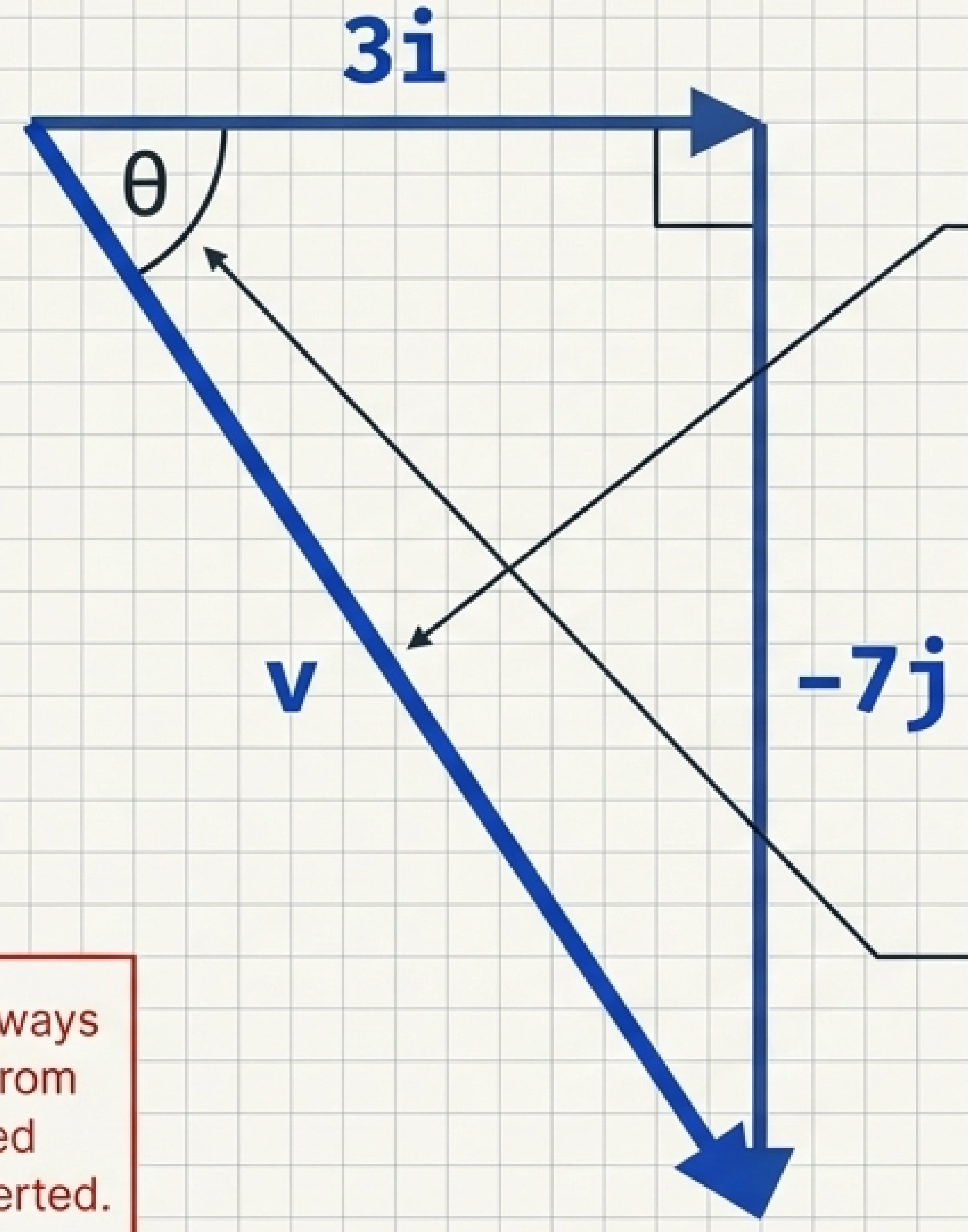
Take care when subtracting negatives.

3. Scaling



Scalar multiplication changes the magnitude but preserves the parallel direction.

Deconstructing a vector yields magnitude and direction



MAGNITUDE |v|

$|v| = \sqrt{(3^2 + (-7)^2} = \sqrt{58} = 7.62$

Magnitude uses Pythagoras' Theorem to yield Speed or Distance.

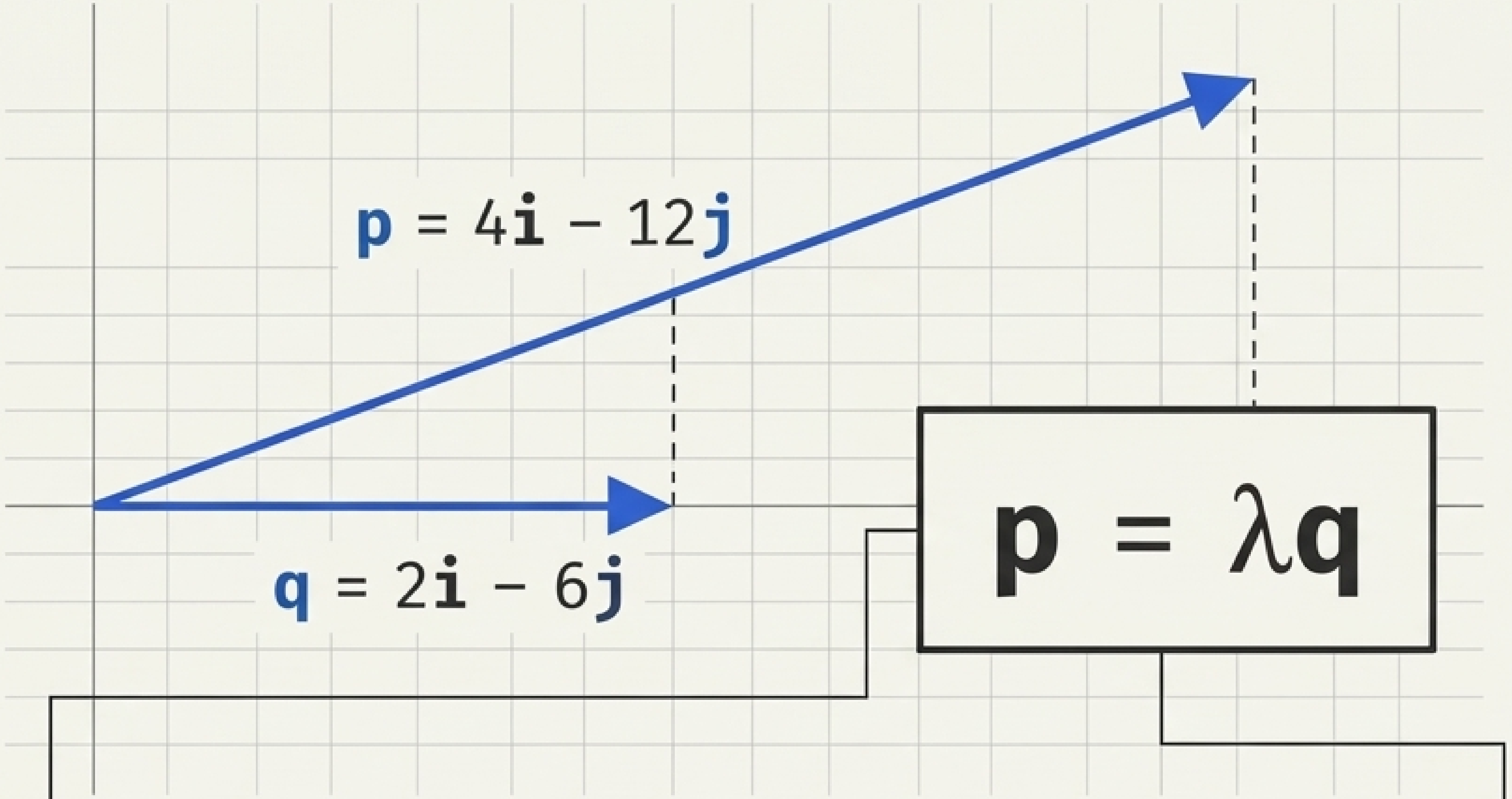
DIRECTION (Angle)

$\tan(\theta) = 7 / 3$

$\theta = 66.8°$

WATCH OUT: Bearings are always measured strictly clockwise from North. A standard angle solved from a triangle must be converted.

Parallel vectors are exact scalar multiples



If vector (**ai** + **bj**) is parallel to vector (**ci** + **dj**), their components share the exact same ratio.

Worked Proof

Scenario:
If vector $(4 + 2\lambda)\mathbf{i} + (3 - 6\lambda)\mathbf{j}$ is parallel to $\mathbf{j}$

Step 1: To be parallel to $\mathbf{j}$, there can be zero horizontal movement.

Step 2: Therefore, the $\mathbf{i}$ component must equal ZERO.

Step 3: $4 + 2\lambda = 0$

Step 4: $\lambda = -2$

Translating motion from scalars to vectors

1D SCALAR KINEMATICS

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

Distance = Speed × Time

($\mathbf{d} = \mathbf{v}\mathbf{t}$)

Speed is simply the magnitude of the velocity vector: $|\vec{\mathbf{v}}|$.

2D VECTOR KINEMATICS

$$\vec{\text{V}}\text{elocity} = \frac{\text{Displacement}}{\text{Time}} \quad \left(\vec{\mathbf{v}} = \frac{\vec{\mathbf{s}}}{\mathbf{t}}\right)$$

Displacement = $\vec{\text{V}}$elocity × Time

($\vec{\mathbf{s}} = \vec{\mathbf{v}}\mathbf{t}$)

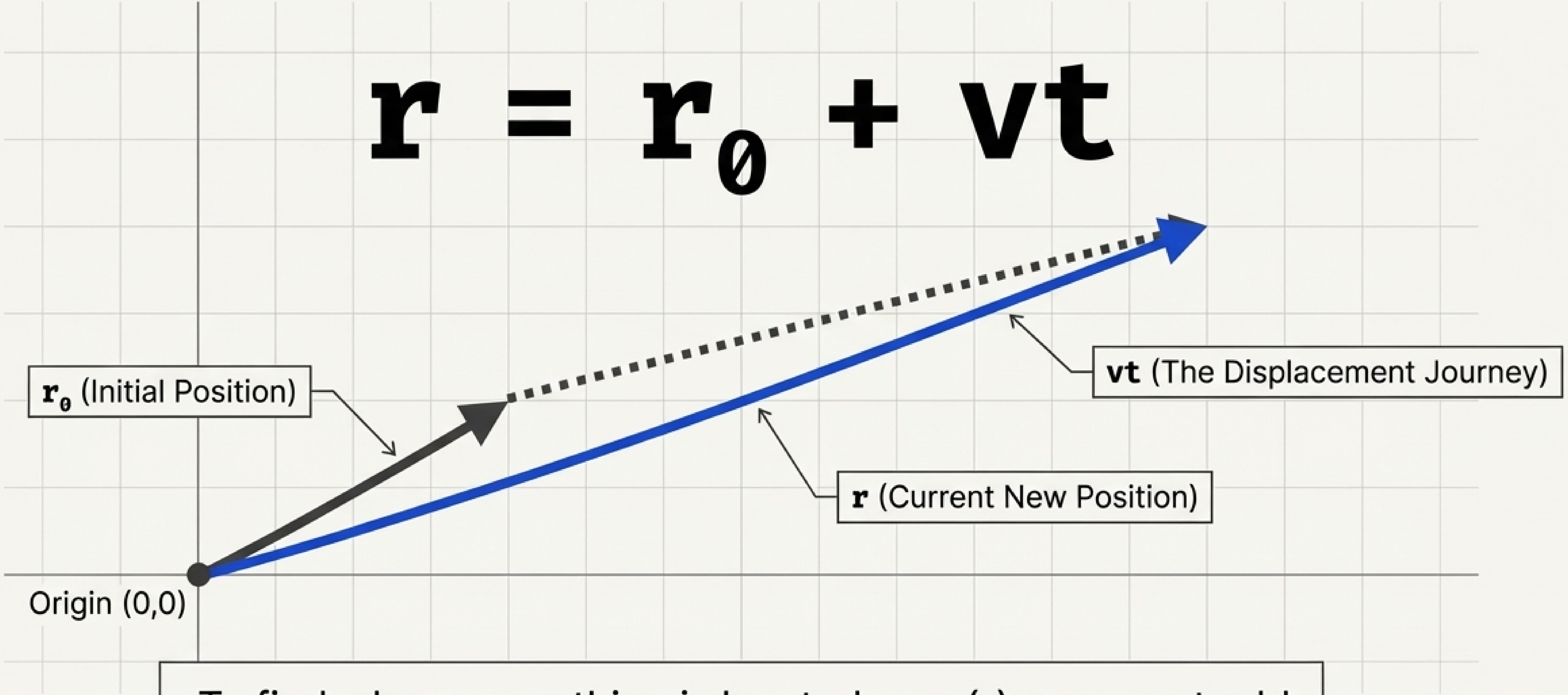
Worked Translation

If a particle moves at constant velocity $\vec{\mathbf{v}} = (3\vec{\mathbf{i}} + \vec{\mathbf{j}})$ m/s for 4 seconds:

Displacement ($\vec{\mathbf{s}}$) $= 4 \times (3\vec{\mathbf{i}} + \vec{\mathbf{j}}) = 12\vec{\mathbf{i}} + 4\vec{\mathbf{j}}$

Distance traveled $= \sqrt{12^2 + 4^2} = \sqrt{160} = 12.6$ m

Tracking exact location with position vectors



To find where something is located now (**r**), you must add where it started (**r_0**) to how far it has traveled (**vt**).

Vector Acceleration and Newton's Second Law

1. Changing Velocity

$$\vec{v} = \vec{u} + \vec{a}t$$

Just as position changes with velocity over time, velocity changes with constant acceleration.

If $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$ and constant $\mathbf{a} = -\mathbf{i} + 2\mathbf{j}$. Find $\mathbf{v}$ at $t=3$.

$\mathbf{v} = (2\mathbf{i} + 3\mathbf{j}) + 3(-\mathbf{i} + 2\mathbf{j})$
$= -\mathbf{i} + 9\mathbf{j}$

2. Applied Force

$$\vec{F} = m\vec{a}$$

A force applied to a particle is a vector. It acts in the exact same direction as the acceleration it produces. Mass (m) acts as the scalar multiplier.

EQUILIBRIUM RULE:
If multiple forces act on a particle in equilibrium, their sum is the zero vector:

$$\sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots = 0\mathbf{i} + 0\mathbf{j}$$

Masterclass 1: Equating Components

Golden Rule: The i components must equal the i components.
The j components must equal the j components.

Problem-Solving Recipe

Scenario & Steps

Scenario: A particle in equilibrium is acted on by forces F_1 = (**ai** − **bj**), F_2 = (**bi** + **aj**), and F_3 = (−4**i** − 2**j**). Find a and b.

Step 1: Write the vector equilibrium equation.

Step 2: Split into simultaneous equations.

Step 3: Solve algebraically.

Execution

$\sum \mathbf{F} = \mathbf{0}$, so $(a + b - 4)\mathbf{i} + (-b + a - 2)\mathbf{j} = \mathbf{0i} + \mathbf{0j}$

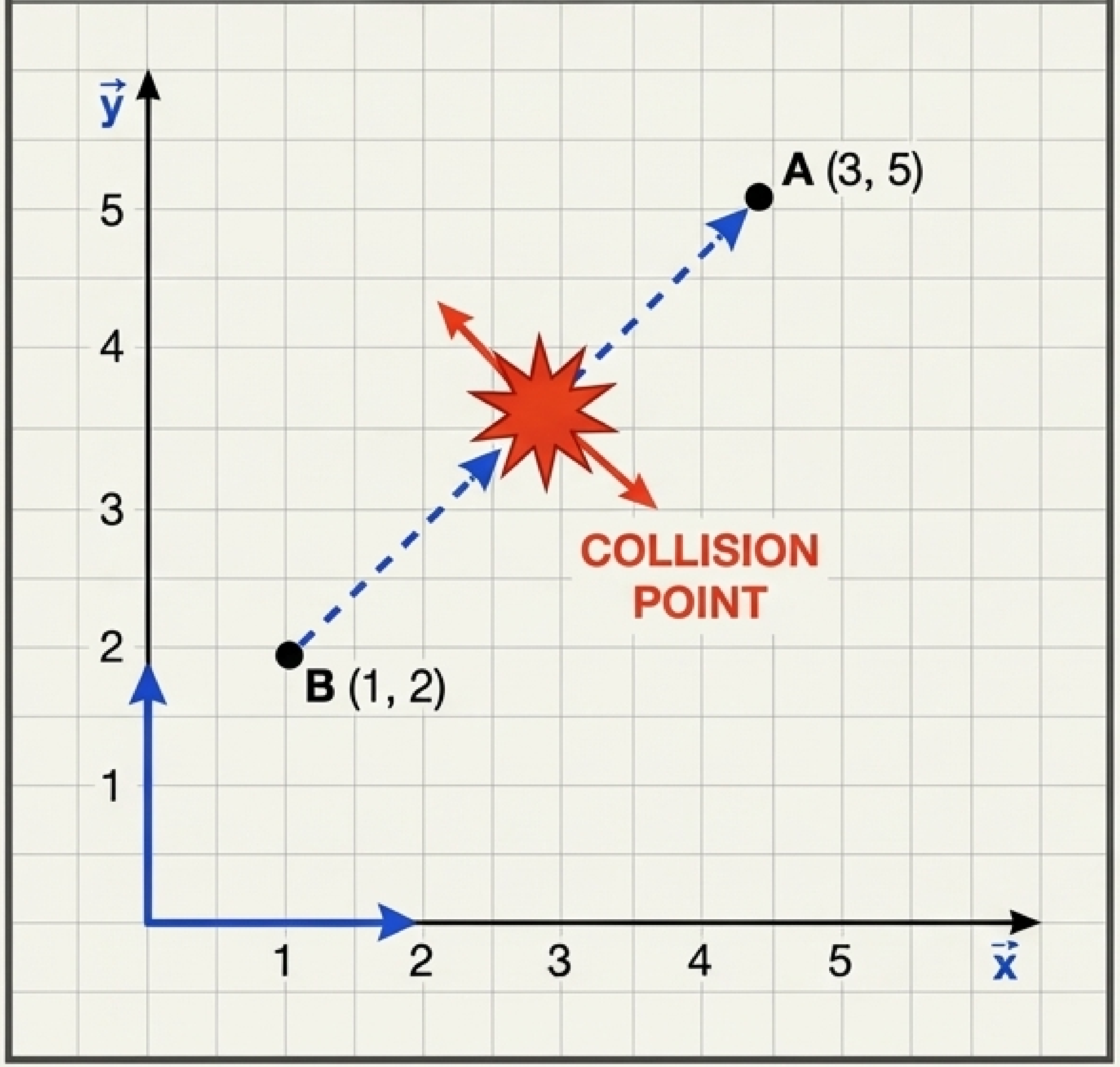
i components: $a + b - 4 = 0$
j components: $a - b - 2 = 0$

Add the equations:
$2a - 6 = 0$ --> **$a = 3$**
Substitute a to find b:
$3 + b - 4 = 0$ --> **$b = 1$**

Masterclass 2: The Collision Blueprint

Editorial Engineering Draft

Intersecting Trajectories



Core Concept Box

For **two particles** to collide, they must have the exact same position vector at the exact same time.

1. Define $\mathbf{r}_A$: Write position vector for A in terms of t.

2. Define $\mathbf{r}_B$: Write position vector for B in terms of t.

e.g., $\mathbf{r}_A = (3 + 2t)\mathbf{i} + (5 - t)\mathbf{j}$

e.g., $\mathbf{r}_B = (1 + 4t)\mathbf{i} + (2 + 2t)\mathbf{j}$

$3 + 2t = 1 + 4t$ --> $2t = 2$ --> $t = 1$

3. Equate Components: Set $\mathbf{r}_A = \mathbf{r}_B$. Equate the **i** terms to find intersec-time.

$\mathbf{r}_A(\mathbf{j}) = 5 - 1 = 4$

$\mathbf{r}_B(\mathbf{j}) = 2 + 2(1) = 4$

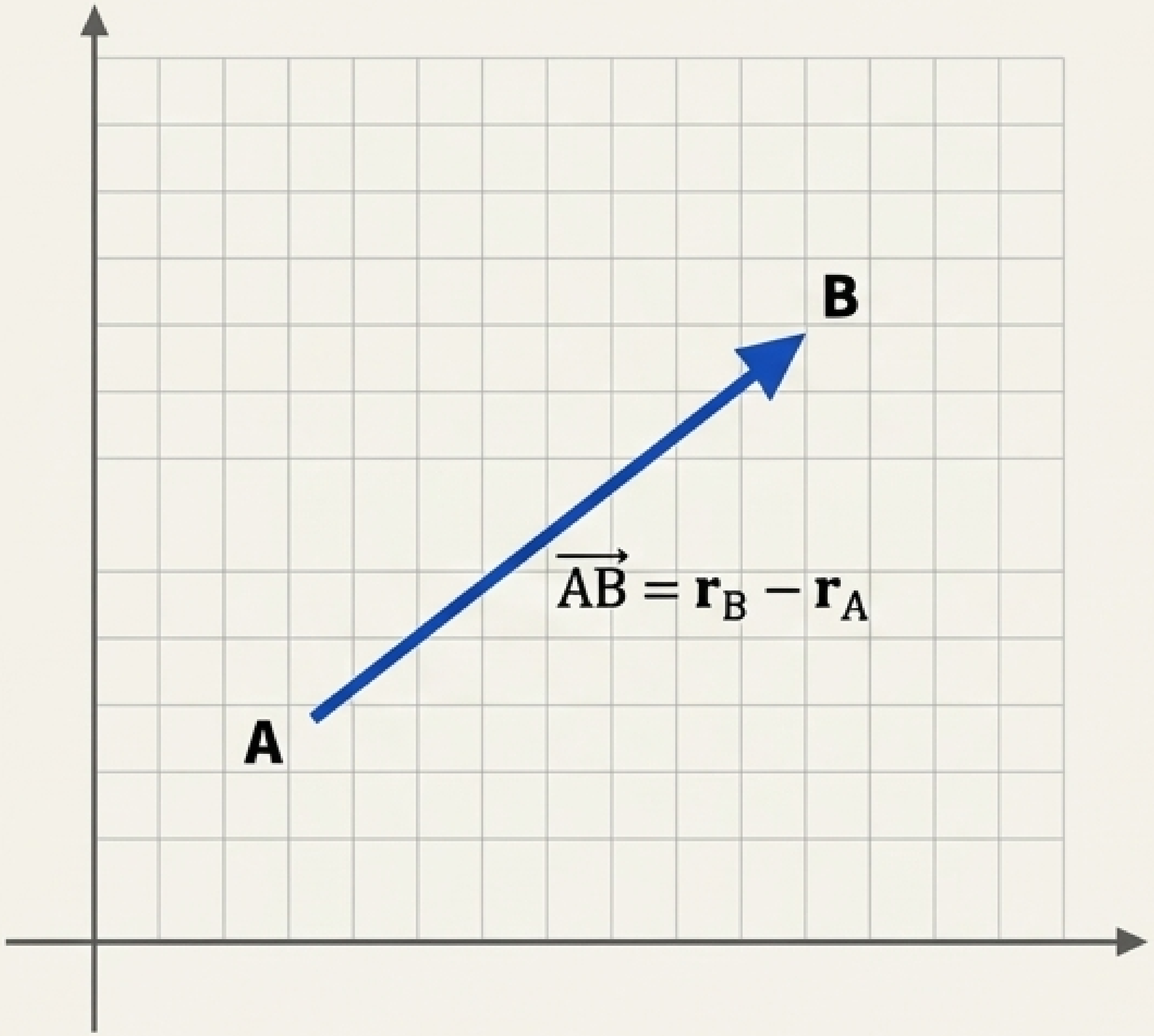
4. Verify: Substitute t into the **j** terms.

If they match (e.g. both equal 4), a collision occurs. If they do not match, their paths cross but at different times.

Masterclass 3: Tracking Relative Distance

Core Concept

To find the distance between two moving objects, you must first find the single vector that connects them at time t.



Workflow Steps: Deriving Distance Squared

1. Find the Connecting Vector:

Subtract their position vectors: Vector $\overrightarrow{AB} = \mathbf{r}_B - \mathbf{r}_A$

Group the resulting $\mathbf{i}$ and $\mathbf{j}$ terms.

e.g., $\overrightarrow{AB} = (10 - 2t)\mathbf{i} + (5 + 3t)\mathbf{j}$

2. Apply Pythagoras for Distance (d):

The distance is the magnitude of the connecting vector: $d = |\overrightarrow{AB}|$

$$d^2 = (10 - 2t)^2 + (5 + 3t)^2$$

3. Expand into a Quadratic:

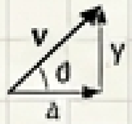
$$d^2 = 13t^2 - 10t + 125$$

PRO-TIP: Exam questions often ask for the *minimum* distance. You can find this by completing the square or differentiating the d^2 quadratic equation!

The Vectors in Mechanics Cheat Sheet

1. GEOMETRY & ALGEBRA

Magnitude: $|\mathbf{v}| = \sqrt{a^2 + b^2}$



Direction: $\tan(\theta) = b/a$ (Always verify the clockwise bearing from North)

Parallel Vectors: $\mathbf{p} = \lambda\mathbf{q}$ (Parallel vectors are exact scalar multiples)

Addition: Add $\mathbf{i}$ terms to $\mathbf{i}$ terms, $\mathbf{j}$ terms to $\mathbf{j}$ terms.

2. KINEMATICS TOOLKIT

Position: $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}\mathbf{t}$

Velocity: $\mathbf{v} = \mathrm{u} + \mathbf{a}\mathbf{t}$

Force: $\mathbf{F} = \mathbf{m}\mathbf{a}$

Displacement: $\mathbf{s} = \mathbf{v}\mathbf{t}$ (when moving at constant velocity)

3. GOLDEN RULES

Speed is the magnitude of velocity: $|\mathbf{v}|$

Distance is the magnitude of displacement: $|\mathbf{s}|$

Collision requires $\mathbf{r}_A = \mathbf{r}_B$ at the exact same time t.

Equilibrium means the Resultant Force sum equals zero: $\sum\mathbf{F} = 0$.